Bounds For Deterministic Identification Capacity in Power-Constrained Poisson Channels

Mohammad J. Salariseddigh

Joint work with:

Uzi Pereg, Holger Boche,
Christian Deppe and Robert Schober

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Outline

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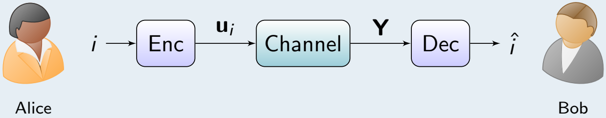
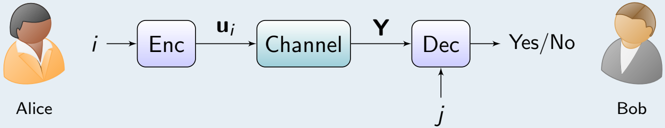
Transmission vs. Identification

- Shannon's setting: Bob recover the message.



- Identification setting: Bob asks if a message was sent or not?



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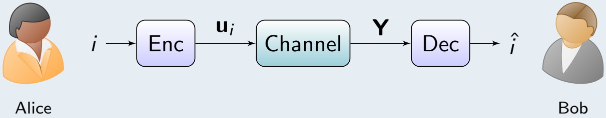

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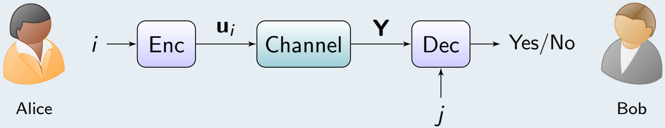

- Molecular communication and healthcare
- Cancer treatment and smart drug delivery
- Any event-triggered scenario

Randomized Identification (RI) [1]

- Originally introduced by Ahlswede and Dueck (1989)
- Capacity was established with randomness at encoder
- Encoder employs distribution to select codewords

Remarkable Property

- Reliable identification is possible with code size growth $\sim 2^{2^{nR}}$
- Sharp difference to transmission with code size growth $\sim 2^{nR}$

[1] R. Ahlswede, and G. Dueck, "Identification via channels", 1989

Deterministic Identification (DI) [2] [3]

- Encoder uses deterministic mapping for coding

Why deterministic?

- Simpler implementation (random resource not required)
- Suitable for Jamming scenarios
- Suitable for molecular communication

[2] R. Ahlswede and N. Cai. "Identification without randomization", 1999

[3] M. J. Salariseddigh, U. Pereg, H. Boche, and C. Deppe, "Deterministic identification over channels with power constraints," IEEE Int'l Conf. Commun. (ICC), 2021 [arXiv:2010.04239, 2021]

Outline

Main Contributions

- We develop lower and upper bounds on the DI capacity for the memoryless discrete time Poisson channels (DTPC) subject to both average and peak power constraints

- We use the bounds to determine the **correct scale**

- We show that the optimal code size scales as $\sim 2^{(n \log n)R}$

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DI Codes

Definition

An $(L(n,R), n, \lambda_1, \lambda_2)$-DI code for DTPC $\mathcal{W}$ is a system $\{(\boldsymbol{u_i}, \mathcal{D}_i)\}_{i\in[1:L(n,R)]}$ subject to

1. Code size: $L(n,R) = 2^{(n \log n)R}$
2. Code-word: $\boldsymbol{u_i} \in \mathcal{X}^n$, decoding sets: $\mathcal{D}_i \subset \mathcal{Y}^n$
3. Input constraints:
   - $0 < u_{i,t} \leq P_{\max}$
   - $n^{-1} \sum_{t=1}^{n} u_{i,t} \leq P_{\text{avg}}$
4. Error requirement type I: $W^n(\mathcal{D}_i | \boldsymbol{u}_i) > 1 - \lambda_1$
5. Error requirement type II: $W^n(\mathcal{D}_i | \boldsymbol{u}_j) \underset{i \neq j}{\leq} \lambda_2$

DI Codes (Cont.)

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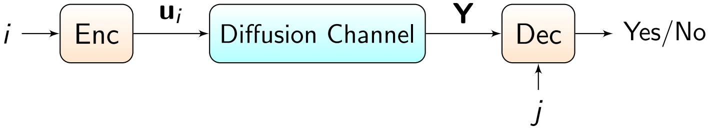
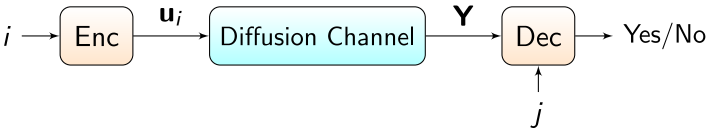
DI for Poisson Channel



- $Y(t) \sim \text{Pois}\left(\lambda + u_i(t)\right)$

DI for Poisson Channel



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Definitions

- Dark current $\rightarrow \lambda \in (0, \infty)$
- Realization of channel output $\rightarrow \mathbf{y} \in \mathbb{N}_0^n$
- Power const. $0 < u_{i,t} \leq P_{\max}$ and $\frac{1}{n}\sum_{t=1}^{n} u_{i,t} \leq P_{\text{avg}}$
- Channel law $\rightarrow W^n(\mathbf{y}|\mathbf{u}_i) = \prod_{t=1}^{n} \frac{e^{-(\lambda+u_{i,t})}(\lambda+u_{i,t})^{y_t}}{y_t!}$

DI for Poisson Channel

Theorem

[4] *Let $\mathcal{W}$ be a DTPC with dark current $\lambda \in (0, \infty)$. Then the DI capacity subject to power constraints $n^{-1} \sum_{t=1}^{n} u_{i,t} \leq P_{avg}$ and $0 < u_{i,t} \leq P_{max}$ for $L(n, R) = 2^{(n \log n)R}$ is bounded by*

$$\frac{1}{4} \leq \mathbb{C}_{DI}(\mathcal{W}, L) \leq \frac{3}{2}$$

[4] arXiv:2107.06061

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Corollary (Traditional Scales)

DI capacity in traditional scales is given by

$$\mathbb{C}_{DI}(\mathcal{W}, L) = \begin{cases} \infty & \text{for } L(n, R) = 2^{nR} \\ 0 & \text{for } L(n, R) = 2^{2^{nR}} \end{cases}$$

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- Achiev. proof: sphere pkg. of rad. $n^{\frac{1}{4}} \Rightarrow 2^{\frac{1}{4}(n \log n)}$ codewords

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Proof Sketch. (Achievability)

- Dense sphere packing arrangement with radius $\sqrt{n\epsilon_n}$
- *Minkowski-Hlawka Theorem* guarantees a density $\Delta \geq 2^{-n}$
- $2^{n \log(n) R} \geq \Delta \cdot \frac{\mathrm{Vol}(\mathcal{Q}_{\mathbf{0}}[n,A])}{\mathrm{Vol}\left(\mathcal{S}_{\mathbf{u}_1}(n,\sqrt{n\epsilon_n})\right)} \geq 2^{-n} \cdot \frac{A^n}{\mathrm{Vol}\left(\mathcal{S}_{\mathbf{u}_1}(n,\sqrt{n\epsilon_n})\right)}$
- $R \geq \frac{1}{n \log n} \left[o(n \log n) + \frac{1}{2} n \log n - \frac{1}{4}(1+b)\, n \log n \right] \xrightarrow{n\to\infty} \frac{1}{4}$

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Chebyshev's inequality leads to the following error bounds:

1. $P_{e,1}(i) \leq \frac{c_1}{n\epsilon_n^2}$
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- Cond. 1 & 2 $\rightarrow \epsilon_n = \frac{A}{n^{\frac{1}{2}(1-b)}}$
  where b is small positive and $A = \min\left(P_{\text{ave}}, P_{\text{max}}\right)$.

Proof Sketch. (Converse)

- We show that if two distinct code-words $\mathbf{u}_i$ and $\mathbf{u}_j$ satisfy $\left|1 - \frac{v_{i_2,t}}{v_{i_1,t}}\right| \leq \epsilon'_n$, for all $t \in [1:n]$, where $v_{i,t} = \lambda + u_{i,t}$ is the letter for shifted codeword, then using the continuity of the Poisson PDF, we obtain

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- We have

$$\left|u_{i_1,t} - u_{i_2,t}\right| = \left|v_{i_1,t} - v_{i_2,t}\right| > \lambda \epsilon'_n$$

- Hence

$$\left\|\mathbf{u}_{i_1} - \mathbf{u}_{i_2}\right\| > \lambda \epsilon'_n$$

Proof Sketch Cont. (Converse)

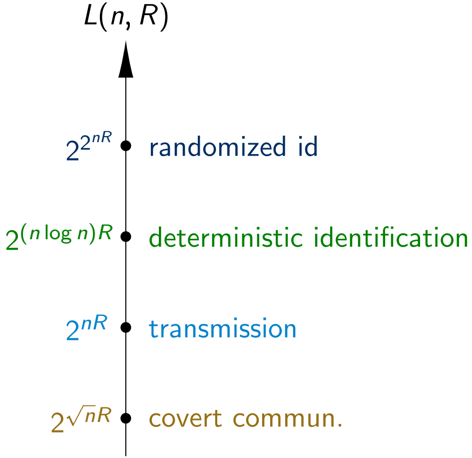
- Tight upper-bound requires:
  1. ϵ'_n large as possible
  2. κ_n tends to zero
- By conditions. 1 & 2 we obtain

$$\epsilon'_n = \frac{P_{\max}}{n^{1+b}}$$

for $b > 0$ being an arbitrarily small

$$\text{rate} \uparrow \iff \epsilon'_n \downarrow$$

Coding Scale



S., Pereg, Boche & Deppe, ITW 2020 [5]

[5] M. J. Salariseddigh, U. Pereg, H. Boche, and C. Deppe, "Deterministic identification over fading channels," IEEE Inf. Theory Workshop (ITW), 2020 [arXiv:2010.10010, 2021]

Outline

Conclusions

- We have determined DI capacity for
  - discrete time Poisson channel $\rightarrow 2^{(n \log n)C} = n^{nC}$ behavior

  As opposed to $2^{2^{nR}}$ for randomized identification

- We observed that DI coding scale is the same for both DTPC and fading channels

- Future directions
  - Address other molecular communication channel models
  - Try Multi-user scenarios

Thank You!