A Study of Capacity and Spectral Efficiency of Fiber Channels

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1) Main Message

An upper bound on the spectral efficiency of a standard optical fiber model

\[ \eta \leq \log(1 + SNR) \text{ [bits/sec/Hz]} \]

- this is the first upper bound on a “full” model
- the bound is tight at low SNR;
- the bound may be extremely loose at high SNR; but it’s better than nothing
2) Information Theory Basics

- **Capacity** $C$ of a channel $P_{Y|X}(.)$ is the maximum $I(X;Y)$ under constraints put on $X$.
- **Example**: real-alphabet additive white Gaussian noise (AWGN) channel
  
  $$Y = X + Z$$
  
  with $\text{Var}[Z] = N$ and an input power constraint $E[X^2] \leq P$ has
  
  $$I(X;Y) \leq C = \frac{1}{2} \log \left( 1 + \frac{P}{N} \right)$$

- Complex alphabet AWGN channels: $C = \log(1+P/N)$
- $N$ is usually taken as $N_0 W$ where $N_0$ is the (one-sided) noise PSD and $W$ is the bandwidth
- **Spectral efficiency** is $\eta = C$ if one uses sinc-pulses of bandwidth $W$
Maximum Entropy

- **Maximum Entropy**: consider $R_X = \mathbb{E}[X X^\dagger]$ where $X$ has length $L$. Then

\[
h(X) \leq \log \left( (\pi e)^L \det R_X \right)
\]

with equality if and only if $X$ is Gaussian and circularly symmetric

- For a complex square matrix $M$ we have

\[
h(MX) = h(X) + 2 \log |\det(M)|
\]

In particular, if $M$ is unitary then $h(MX) = h(X)$
Entropy Power Inequality

- Entropy Power:
  \[ V(X) = e^{h(X)/L} / (\pi e) \]

- Entropy Power Inequality: for independent \( X \) and \( Y \) we have
  \[ V(X + Y) \geq V(X) + V(Y) \]

- Conditional version: for conditionally independent \( X \) and \( Y \) we have
  \[ V(X|U) = e^{h(X|U)/L} / (\pi e) \]
  \[ V(X + Y|U) \geq V(X|U) + V(Y|U) \]
3) Fiber Channel(s)

• To simulate, split the fiber length $z^*$ into $K$ small steps ($\Delta z$) and the time $T$ into $L$ small steps ($\Delta t$)

• Split-step Fourier method at distance $z_k$, $k=0,1,...,K$

• Ideal Raman amplification: removes the loss but adds noise

• $F =$ Fourier transform

• $D_L =$ diagonal matrix with fixed entries of unit amplitude (all-pass filter)

• $D_N =$ diagonal matrix with unit amplitude entries; the $(\ell,\ell)$-entry phase shift is proportional to the magnitude-squared of the $\ell$th entry of $E_N(z_{k+1})$
4) An Upper Bound

Main Observations
- The linear step conserves energy and entropy
- The non-linear step also conserves energy and entropy

\begin{align*}
  h\left(|a| e^{j \arg(a)} + jf(|a|)\right) &= h(|a|, \arg(a) + f(|a|)) + E[\log|a|] \\
  &= h(|a|) + h\left(\arg(a) + f(|a|) \mid |a|\right) + E[\log|a|] = h(a)
\end{align*}
Energy Recursion

- Energy after K steps: \(E_{\text{Launch}} + KN\). We thus have:

\[
\begin{align*}
    h(E(z_K)) &\leq \log \left( (\pi e)^L \det \left( R(E(z_K)) \right) \right) \quad \text{… maximum entropy} \\
    &\leq \sum_{i=1}^{L} \log \left( \pi e R_{i,i}(E(z_K)) \right) \quad \text{… Hadamard's inequality} \\
    &\leq L \cdot \log \left( \pi e (E_{\text{Launch}} + KN)/L \right) \quad \text{… Jensen's inequality}
\end{align*}
\]
Entropy recursion:

\[
V\left( E\left( z_{k+1}\right) | E\left( z_0\right) \right) \geq V\left( E\left( z_k\right) | E\left( z_0\right) \right) + N/L
\]

We thus have:

\[
V\left( E\left( z_K\right) | E\left( z_0\right) \right) \geq KN/L
\]

or

\[
h\left( E\left( z_K\right) | E\left( z_0\right) \right) \geq L \log(\pi e KN/L)
\]
So for every step we have:

- **Signal energy** grows by the noise variance: can **upper** bound \( h(\mathbb{E}(z_k)) \)
- **Entropy power** grows by at least the noise variance: can **lower** bound \( h(\mathbb{E}(z_k) \mid \mathbb{E}(z_0)) \)
- Result*:

\[
I(\mathbb{E}(z_0);\mathbb{E}(z_K)) = h(\mathbb{E}(z_K)) - h(\mathbb{E}(z_K) \mid \mathbb{E}(z_0)) \\
\leq L \cdot \log(1 + SNR)
\]

*SNR = receiver signal-to-noise ratio
\[
\Rightarrow \quad \frac{1}{L} I(E(z_0); E(z_K)) \leq \log(1 + \text{SNR})
\]

- Let \( B = 1/\Delta t \) be the “bandwidth” of the simulation
- So \( L = T/\Delta t = TB \) is the time-bandwidth product
- The spectral efficiency is thus bounded by

\[
\eta \leq \log(1 + \text{SNR}) \quad \text{[bits/sec/Hz]}
\]
Q1: Why normalize by the simulation bandwidth $B$? The “real” bandwidth $W$ can be smaller.
A1: $B$ can be chosen (this is even desirable) as the smallest bandwidth for which simulations give accurate results.

Q2: What about capacity?
A2: Any real fiber has a maximal bandwidth $B_{\text{max}}$. A capacity upper bound follows by multiplying $B_{\text{max}}$ by $\log(1+\text{SNR})$.

$$\eta \leq \log(1+\text{SNR}) \quad \text{[bits/sec/Hz]}$$
Q3: What about MIMO fiber?
A3: If energy is preserved by the linear and non-linear steps, and the noise is AWGN then the above bound remains valid per mode

Q4: What about frequency-dependent (or mode-dependent) loss?
A4: Open research!

Q5: What about lower bounds?
A5: Apply entropy recursion to $V( E(z_k) )$ and energy recursion to $h( E(z_k) | E(z_0) )$. Issues (looks solvable): bandwidth expansion bounds

$$\eta \leq \log(1 + SNR) \quad [\text{bits/sec/Hz}]$$
Conclusions

1) Nonlinear cascade models are fun to study ... many other applications
2) Spectral efficiency of SMF with linear polarization is \[ \leq \log(1+\text{SNR}) \]
3) Many extensions are possible:
   - lumped amplification, 3\textsuperscript{rd}-order dispersion, delayed Kerr effect
   - uniform loss, linear filters (for capacity results)
   - MIMO fiber (MMF or MCF) if the linear and non-linear steps conserve energy and entropy, and the noise is Gaussian and white
4) More difficult:
   - better bounds and understanding at high SNR
   - frequency-dependent loss, dispersion, non-linearity
5) Network information theory for fiber should be developed