Design of Coarsely-Quantized Message Passing Decoders

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Binary Message Passing (BMP) for LDPC Codes

- Kou, Lin, Fossorier, LDPC codes based on finite geometries, IEEE Trans. IT, 2001
- Miladinovic, Fossorier, Improved bit-flipping decoding, IEEE Trans. IT, 2005
- Ardakani, Kschischang, Properties of binary message-passing, IEEE Trans. IT, 2005
- Sankaranarayanan et al., Failures of the Gallager B decoder, ISIT 2006
- Planjery, Declercq, Danjean, Vasic, Finite alphabet iterative decoders, 2013-
- Many other papers

Here Review and Expand on:

I. Low-Density Parity-Check (LDPC) Codes

- A binary linear block code is the set of binary (row) vectors, or codewords, \( \mathbf{c} \), satisfying, e.g.,

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{c} \\
\end{bmatrix} = 0
\]

where \( H \) is a \((n-k) \times n\) parity-check-matrix. Rate is \( R = 1 - \text{rank}(H)/n \) (example: \( R = 5/8 \)).

Tanner Graph Representation of Parity-Check Constraints

Variable Nodes

Edge Interleaver

Check Nodes

degree \( d_v = 2 \)

degree \( d_c = 4 \)
- Code is **low-density** if each row and column of $H^T$ has “few” 1’s
- **Irregular** LDPC code: variable number of 1’s in every column/row
- Decoding: use message passing on the graph
- Messages may be cond. probabilities
  $$\Pr\left( c_1 = 0 \mid y \right)$$
  or log-likelihood ratios (L-values)
  $$L_1 = \log \frac{\Pr\left( c_1 = 0 \mid y \right)}{\Pr\left( c_1 = 1 \mid y \right)}$$
  or, in practice, quantized L-values
II. Iterative Decoding

LDPC code decoder iterations (turbo processing):

- a-priori information
- extrinsic information
- decoder computations performed here
III. Demodulation and Decoding

- L-values are real but must be quantized, see figure below
  1) Demodulator: can put out soft decisions ($>\log_2(M)$ bits/symbol) or hard decisions ($=\log_2(M)$ bits/symbol)
  2) Decoder iterations: B-bit message passing: binary message passing (BMP, $B=1$) ternary message passing (TMP, $B \approx 2$)

- Motivation: high-speed devices ($>100$ Gb/s) need simplifications
**BMP/TMP**: natural approaches are as follows:

1) Every edge bit represents a hard decision on an extrinsic L-value.
2) Variable nodes **convert** apriori bits to L-values, **add** L-values, and **make** binary (hard) or ternary decisions on output L-values.
3) Check nodes perform (extrinsic) XORs for **binary** message passing, and (extrinsic) XORs and erasures for **ternary** message passing.

**Analysis**: use **distribution evolution (DE)** to track extrinsic probabilities. **BMP**: track error probabilities; **TMP**: also track erasures.

\[
\begin{align*}
L_{ch} & \quad y \\
\epsilon_{av} & \quad \epsilon_{ac} \\
\epsilon_{ev} & \quad \epsilon_{ec} \\
\end{align*}
\]

N.B. \(\epsilon_{ec}(\epsilon_{ac})\) depends on \(d_c\).
Check node (degree $d_c$) and binary messages:

$$
\epsilon_{ec} = f_c(\epsilon_{ac}; d_c) = \frac{1 - (1 - 2\epsilon_{ac})^{d_c-1}}{2}
$$

Variable node (degree $d_v$): suppose $x_j = \pm 1$ (BPSK)

$$
\epsilon_{ev} = \sum_{j=1}^{d_v} \frac{1}{d_v} \Pr \left[ \text{sgn}(L_{ev,j}) \neq x_j \right]
$$

$$
L_{ev,j} = L_{ch} + \sum_{i=1; i \neq j}^{d_v} L_{av,i}, \quad j = 1, 2, \ldots, d_v
$$

$$
L_{av,i} = a_i \log \frac{1 - \epsilon_{av}}{\epsilon_{av}}, \quad a_i = \pm 1, \text{ but what is } \epsilon_{av}?
$$

$L_{ch}$ depends on channel quantization

Two design issues
**Issue 1: Variable Node Processing**

- Processing depends on $\epsilon_{av}$ which
  - Varies from iteration to iteration
  - Is *unknown*, unless the codes have infinite length in which case $\epsilon_{av}$ can be computed from EXIT chart (see below)

- Two other approaches:
  - Optimize “choice” of $\epsilon_{av}$ offline by numerical simulation
  - Estimate $\epsilon_{av}$ online based on the number of unsatisfied checks

- 1\textsuperscript{st} approach is complex, but likely very good. This variant was used to design certain deployed LDPC codes

- 2\textsuperscript{nd} approach is used here
Issue 2: Channel Outputs

- Consider an AWGN channel, $x_j=\pm 1$, noise variance $\sigma_n^2$
- Let $D_{ch}=|L_{ch}|$ ... called the reliability of the L-value
- For soft decisions:
  $$D_{ch} = \frac{2}{\sigma_n^2}$$
- For hard decisions get a binary symmetric channel (BSC) with crossover probability $\varepsilon_{ch}$ ($0 \leq \varepsilon_{ch} \leq 0.5$)
  $$D_{ch} = \log \frac{1 - \varepsilon_{ch}}{\varepsilon_{ch}}$$, where $\varepsilon_{ch} = Q\left(1/\sigma_n\right)$
- For b-bit quantization: use mixture of b hard-decision channel reliabilities, e.g., 2-bit quantization with a binary symmetric quaternary output (BSQC) channel
Channel: $\sigma_n = 0.67$

- **Variable nodes** on the y-axis to the x-axis:
- **Check nodes** on the y-axis to the x-axis:

$$I_{ac} = 1 - h(\epsilon_{ac})$$
where $h(x)$ is the binary entropy function:

$$h(x) = -x \log_2 x - (1-x) \log_2 (1-x)$$

Example: $h(0.11) = 0.5$

Similar for $I_{ec}$, $I_{av}$, $I_{ev}$

**Comments:**
- BSC quantization same as Gallager B algorithm
- BSQC quantization thresholds at 0 and ±1.9
IV. Optimization: Irregular LDPC Codes

- Each node’s $\epsilon_{ev}$ depends on $d_v$: write as $\epsilon_{ev}(\epsilon_{av},d_v)$. Now use different degrees to shape avg. variable node curve:

$$
\epsilon_{ev}(\epsilon_{av}) = \sum_{i} \lambda_i \epsilon_{ev}(\epsilon_{av},i)
$$

with $\lambda_i=$fraction of edges connected to var. nodes of degree $i$

- Can similarly shape the check node function $\epsilon_{ec}(\epsilon_{ac})$

- Degree distribution $\{\lambda_i\}$ design: use EXIT chart
  - $\epsilon_{ev}(\epsilon_{av})$ curve should lie above $\epsilon_{ec}(\epsilon_{ac})$ curve for convergence (and $n=\infty$)
  - L-value messages: Matching EXIT curves maximizes rate.

- BMP: new issues vs. L-value messages
  - Stability (decoder convergence when $\epsilon_{av}$ or $\epsilon_{ac}$ are small)
  - Cycles related to “absorbing sets” cause decoder to get stuck

- Approach: build optimization & remedies into a linear program
Rate, Stability, Cycles

- **Design Rate:**
  \[ R = 1 - \frac{1/d_c}{\sum_{i} \lambda_i / i} \]

- **Stability:** satisfied for binary message passing and hard or soft channel messages if and only if (try \( \lambda_2 = 1 \))
  \[ (\lambda_2 + 2\varepsilon_{ch}\lambda_3)(d_c - 1) < 1 \]

- **Cycles:**
  - Structure on right causes decoding failure if all channel messages in error, and if all other incoming messages correct
  - Obvious idea: avoid cycles of degree 2 or 3 variable nodes
Result: a Tanner graph with no cycles having degree 2 and 3 variable nodes exists if and only if (try $\lambda_3 = 1$)

$$3\lambda_2 + 4\lambda_3 \leq \frac{6}{d_c}\left(1 - \frac{1}{(1 - R)N}\right) < \frac{6}{d_c}$$

Linear Program: $\lambda = \{\lambda_i\}$ is variable node degree distribution

$$\hat{\lambda}^* = \arg\max_\lambda R = \arg\max_\lambda \left(1 - \frac{1/d_c}{\sum_i \lambda_i/i}\right) = \arg\max_\lambda \sum_i \lambda_i/i$$

subject to [variable node EXIT curve above check node EXIT curve]

$$\sum_i \lambda_i = 1, \quad 0 \leq \lambda_i \leq 1$$

$$(\lambda_2 + 2\varepsilon_{ch}\lambda_3)(d_c - 1) < 1, \quad 3\lambda_2 + 4\lambda_3 < \frac{6}{d_c}$$
BMP Thresholds

Comments:
- x-axis is $E_s/N_0$
- hard decision (BSC) capacity $\approx 2$dB below AWGN capacity at low rate
- gap to capacity decreases as rate increases, for hard decisions and BMP
- Conclusion: high rate is good for BMP

Binary message passing & variable node degree distribution optimized using linear program
Performance: Rate 1/2, BMP

Comments:
- $x$-axis is $E_b/N_0$
- PEG interleavers automatically avoid undesirable cycles
- $n = 10,000$
- 2-bit quant. gains
  $\approx 1$ dB over Gallager B and loses $\approx 0.5$ dB as compared to soft outputs

Hard-Decision Cap: 1.8 dB
Soft-Decision Cap: 0.2 dB

Optimized Codes
**Performance: Rate 15/16, BMP**

- **Hard-Decision Capacity**: 5.0 dB
- **Soft-decision Capacity**: 3.9 dB

**Comments:**
- x-axis is $E_b/N_0$
- interleaver taken from standard
- 2-bit quant. gains
  - $\approx 1$ dB over Gallager B and loses $\approx 0.2$ dB vs. soft outputs
- BMP is $\approx 1.5$ dB from L-value
- message capacity
- longer & irregular codes get closer

regular LDPC code for optical
see ITU-T G.975.1 2004
Comments:
- Figure taken from Emna Ben Yacoub’s Master Thesis, Oct. 2018
- Curves show decoding thresholds with BMP and TMP for optimized protograph LDPC code ensembles
Fig. 2. FER versus $E_b/N_0$ for TMP and unquantized BP decoding for $R = 3/4$ (---), $R = 5/6$ (---) and $R = 7/8$ (---). We compare the TMP performance of optimized codes (---) to their AR4JA counterparts with unquantized BP (---) and TMP decoding (---).
For More Details:


See the Posters!
And the First Talk Tomorrow!