One and Two Bit Message Passing Decoding for SC-LDPC Codes and Higher-Order Modulation

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Introduction

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• Data flow within an LDPC decoder\(^1\):

\[
F_{\text{LDPC}} = \frac{2 \cdot n_c \cdot D \cdot q \cdot d_{\text{avg}}}{R_c}
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• Data flow within an LDPC decoder:\(^1\):

\[
F_{LDPC} = \frac{2 \cdot n_c \cdot D \cdot q \cdot d_{avg}}{R_c}
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• Well, but why shouldn’t we still exploit some of the soft-information?

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One and Two Bit Message Passing
Binary Message Passing (BMP) (I)$^2$

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Binary Message Passing (BMP) (I)²

Variable node update

\[ m_{V_i \rightarrow C_j}^{(\ell)} = \begin{cases} +1, & \text{if } \left( \sum_{j' \in N(C_j)} D^{(\ell)} \cdot m_{C_j' \rightarrow V_i}^{(\ell)} \right) + L_i > 0 \\ -1, & \text{if } \left( \sum_{j' \in N(C_j)} D^{(\ell)} \cdot m_{C_j' \rightarrow V_i}^{(\ell)} \right) + L_i < 0 \end{cases} \]

Binary Message Passing (BMP) \(^{(1)}\)^2

Check node update

\[
m_{C_j \rightarrow V_i}^{(\ell)} = \prod_{i' \in N(C_j)} m_{V_{i'} \rightarrow C_j}^{(\ell)} \quad \in \{-1, +1\}
\]

---

Binary Message Passing (BMP) (II)

- Extrinsic channel is a binary symmetric channel (BSC).
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• Weighting factor $D^{(\ell)}$ for each iteration is calculated off-line via density evolution and stored.
Ternary Message Passing (TMP) (I)
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Variable node update

\[
m_{V_i \rightarrow C_j}^{(\ell)} = \begin{cases} 
+1, & \text{if } \left( \sum_{j' \in N(V_i) \setminus \{j\}} D^{(l)} \cdot m_{C_{j'} \rightarrow V_i}^{(\ell)} + L_i \right) > a \\
-1, & \text{if } \left( \sum_{j' \in N(V_i) \setminus \{j\}} D^{(l)} \cdot m_{C_{j'} \rightarrow V_i}^{(\ell)} + L_i \right) < -a \\
0, & \text{else.}
\end{cases}
\]
Ternary Message Passing (TMP) (I)

Check node update (same as BMP)

\[ m^{(\ell)}_{C_j \rightarrow V_i} = \prod_{i' \in \mathcal{N}(C_j) \setminus \{i\}} m_{V_{i'} \rightarrow C_j} \]
Ternary Message Passing (TMP) (II)

• Region between $-\alpha$ and $\alpha$ is declared to be an erasure region (complete uncertainty).

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\[ x \in [-a, a] \]

- Extrinsic channel is a binary error and erasure channel (BEEC).
- Straightforward to use with punctured LDPC codes or for channels with erasures (strong fading).
Quaternary Message Passing (QMP) (I)

- TMP already requires two bits. Why not introduce another level?
- We devide the real line into four regions:

```
| -H | -L | +L | +H |
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Extrinsic channel is a symmetric quaternary output channel.
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  \[ x \]

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Quaternary Message Passing (QMP) (II)
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Variable node update

\[ m^{(\ell)}_{V_i \rightarrow C_j} = \begin{cases} 
-H, & \left( \sum_{j' \in \mathcal{N}(V_i) \setminus \{j\}} D^{(l)} \cdot m^{(\ell)}_{C_{j'}, \rightarrow V_i} + L_i \right) \leq -a \\
-L, & -a < \left( \sum_{j' \in \mathcal{N}(V_i) \setminus \{j\}} D^{(l)} \cdot m^{(\ell)}_{C_{j'}, \rightarrow V_i} + L_i \right) < 0 \\
+L, & 0 \leq \left( \sum_{j' \in \mathcal{N}(V_i) \setminus \{j\}} D^{(l)} \cdot m^{(\ell)}_{C_{j'}, \rightarrow V_i} + L_i \right) < a \\
+H, & \left( \sum_{j' \in \mathcal{N}(V_i) \setminus \{j\}} D^{(l)} \cdot m^{(\ell)}_{C_{j'}, \rightarrow V_i} + L_i \right) \geq a 
\]
Quaternary Message Passing (QMP) (II)

Check node update (classical min-sum)

\[
m^{(\ell)}_{C_j \rightarrow V_i} = \min_{V_i' \in \mathcal{N}(C_j) \setminus \{i\}} |m^{(\ell-1)}_{V_i' \rightarrow C_j}| \times \prod_{V_i' \in \mathcal{N}(C_j) \setminus \{i\}} \text{sign} \left( m^{(\ell-1)}_{V_i' \rightarrow C_j} \right)
\]
Higher Order Modulation
System Model

- Additive white Gaussian noise channel with $N_i$ iid. $\mathcal{N}(0, \sigma^2)$.

$$Y_i = X_i + N_i, \quad i = 1, \ldots, n.$$ 

- Discrete signaling with constellation $\mathcal{X}$: $M = 2^m$-ASK.

- Binary reflected Gray code (BRGC) labeling.
- Mapping of constellation point $x \in \mathcal{X}$ to its label via

$$b = (b_1 b_2 \ldots b_m) = \chi(x) \in \{0, 1\}^m.$$
Bit-Metric Decoding

- Bit-metric, soft-decision (SD) decoders:

\[ l_j(y) = \log \frac{P_{Bj|Y}(0|y)}{P_{Bj|Y}(1|y)}, \quad j = 1, \ldots, m. \]
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- The reliability of each bit-level is different.
- The PDF of the soft-information is not symmetric, i.e.,

\[ p_{L|B}(l|0) \neq p_{L|B}(-l|1). \]
Structured Ensembles: Protographs

- **Structured LDPC codes** (e.g., protograph-based, multi-edge type) are particularly suited for the optimization with different reliabilities.

\[
B = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}
\]

Final parity-check matrix derived via (cyclic) lifting operation.
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- Protograph-based LDPC codes: Defined via **small basematrix** $B \in \{0, 1, \ldots, S\}^{m_P \times n_P}$.

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- For higher-order modulation and BMD, this does not hold in general.
- Use channel adapters:\(^3\)

\[
\tilde{L}_j = L_j \cdot (1 - 2B_j).
\]

---

Density Evolution for BMP/TMP/QMP (II)

Track the evolution of the following probabilities for all edges $[B]_{ij} \neq 0, i \in \{1, \ldots, m_P\}, j \in \{1, \ldots, n_P\}$. 

- **BMP** to CN: $p(\ell-1)_{ij}$
  - VN to CN: $q(\ell-1)_{ij}$
- **TMP** to CN: $p(\ell-1)_{ij}$
  - VN to CN: $q(\ell-1)_{ij}$
- **QMP** to CN: $p(\ell-1)_{ij}$, $p(\ell-1)_{L_{ij}}$ (for QMP)
  - VN to CN: $q(\ell-1)_{ij}$, $q(\ell-1)_{L_{ij}}$, $p(\ell-1)_{L_{ij}}$ (for QMP)
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- **QMP**
  - VN to CN: \(p_{-H}^{(\ell)}(i, j), p_{-L}^{(\ell)}(i, j), p_L^{(\ell)}(i, j)\)
  - CN to VN: \(q_{-H}^{(\ell)}(i, j), q_{-L}^{(\ell)}(i, j), p_L^{(\ell)}(i, j)\)
Input Parameters for DE (I)

- We face $m$ different bit-channels $p_{L_i|B_i}, i = 1, \ldots, m$ with BMD.
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• Example: 8-ASK, \( m = 3 \) bit channels.
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\[
p_0^{(\ell)}(i, j) = \Pr \left\{ -a \leq \tilde{L}_{\text{ch}, T(j)} + L_{\text{in}}^{(\ell)} \leq a \right\}
\]

\[
= \sum_z \Pr \left\{ L_{\text{in}}^{(\ell)} = z \right\} \int_{-a-z}^{a-z} p_{\tilde{L}_{T(j)}}(\ell) \, d\ell
\]

\[
p_{-1}^{(\ell)}(i, j) = \Pr \left\{ L_{\text{ch}} + L_{\text{in}}^{(\ell)} < -a \right\}
\]

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\]

- Full description of DE for BMP/TMP/QMP in papers\(^4\)\(^5\).

---


Obtaining the CDFs of the soft-information

We propose two simple approaches.

1. Monte Carlo based estimation of the CDF.

Exemplary calculation for TMP erasure region ($\mu_{\text{ch},j} = 2/\tilde{\sigma}^2_j$ and $\sigma^2_{\text{ch},j} = 4/\tilde{\sigma}^2_j$):

$$p_0(i,j) = Q\left(\frac{-a + \mu_{\text{ch},T}(j)}{\sigma_{\text{ch},T}(j)}\right) - Q\left(\frac{a + \mu_{\text{ch},T}(j)}{\sigma_{\text{ch},T}(j)}\right).$$
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2. Surrogate approach: Find equivalent AWGN surrogate \( \tilde{Y}_j = \tilde{X}_j + \tilde{N}_j \) with \( \tilde{X}_j \in \{-1, +1\} \) and \( \tilde{N}_j \sim \mathcal{N}(0, \tilde{\sigma}_j^2) \) with

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CDF Plot: 4-ASK uniform
Numerical Results
Setup for the Numerical Examples

• We target a spectral efficiency of $\approx 1.5 \text{ bpcu}$. 

• Code class: SC-LDPC codes with VN degree four.
  • Based on protographs: $[4 \ 4 \ 4]$, $[4 \ 4 \ 4 \ 4]$, $[4 \ 4 \ 4 \ 4 \ 4 \ 4]$.
  • Constructed via edge spreading, therefore memory 3.
  • Thresholds: Right unterminated and window decoding with $W = 15$.
  • Finite length: Terminated after $L = 50$ spatial positions.
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Numerical Results: Asymptotics

Achievable rates (Unconstrained Shannon limit: 8.45 dB):

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<td>9.50</td>
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Numerical Results: Finite Length 4-ASK uniform
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![Graph showing FER vs. SNR for different decoding methods.](image)
Numerical Results: Finite Length 4-ASK uniform

![Graph showing FER vs SNR for different decoding methods: Full BP, BMP, TMP, QMP. The graph plots SNR in dB on the x-axis and FER on the y-axis, with distinct markers for each decoding method at various SNR values.]
Numerical Results: Finite Length 4-ASK uniform

![Graph showing FER vs SNR for different decoding methods]

- Full BP
- BMP
- TMP
- QMP

SNR [dB] vs FER with markers indicating 0.75 dB and 0.82 dB improvements.
Numerical Results: Finite Length 8-ASK PS

![Graph showing FER vs SNR for different decoding methods. The graph plots FER (false error rate) on a logarithmic scale against SNR (signal-to-noise ratio) in dB. The x-axis ranges from 8.5 to 11 dB, and the y-axis ranges from $10^{-6}$ to $10^0$. The graph includes lines for Full BP, BMP, TMP, and QMP, with markers indicating specific performance points. At 9.5 dB, the Full BP curve shows a significantly lower FER compared to the other methods, with a gap of approximately 0.76 dB between Full BP and the BMP, and 0.77 dB between Full BP and the QMP.]
Conclusions

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